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Generalized coordinate for warping of naturally curved and twisted beams with general cross-sectional shapes

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Abstract

This paper presents an efficient procedure for analyzing naturally curved and twisted beams with general cross-sectional shapes. The hypothesis concerning the cross-sectional shapes of the beams is abandoned in this analysis, and relatively general equations are derived for the analysis of such a structure. Solving directly such equations for various boundary conditions, which take into account the effects of torsion-related warping as well as transverse shear deformations, can yield the solutions to the problem. The solutions can be used to calculate various internal forces, stresses, strains and displacements of the beams. The present theory will be used to investigate the stresses and displacements of a cantilevered curved beam subjected to action of arbitrary load. The numerical results are very close to the FEM results.

Keywords: Naturally curved and twisted beam; St. Venant torsional warping function; Generalized coordinate for warping; The minimum potential energy principle

1. Introduction

Static and dynamic analysis of naturally curved and twisted beams has many important applications in mechanical and civil engineering. The problem is urgently needed to be thoroughly studied in engineering structures, especially in bridge structures associated with curved beams. Washizu (1964) presented an approximate theory of the beams and derived a system of governing equations. The unknowns in these equations are the displacement components $(u_s, u_\xi, u_\eta, \varphi_s, \varphi_\xi, \varphi_\eta)$ together with a generalized warping

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coordinate (α). The solutions of these equations, however, were not obtained. The work of Aimin et al. (2002), developed such a theory and provided such an example where the beams are considered to have a double symmetric cross-section, a fact which clearly restricts the applicability of these equations.

This paper aims to derive a differential equation for generalized warping coordinate α which involves the St. Venant torsional warping function, the equation applicable to the case of the beams with general cross-sectional shapes subjected to arbitrary load. Obviously, such a system of equations is of practical use for a variety of engineering applications.

Let the locus of the cross-sectional centroid of a beam be a continuum curve in space, the tangential, normal and bi-normal unit vectors of the curve l are $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$, respectively. The Frenet–Serret formulae, for a smooth curve, is:

$$\boldsymbol{t}' = k_1\boldsymbol{n}, \quad \boldsymbol{n}' = -k_1\boldsymbol{t} + k_2\boldsymbol{b}, \quad \boldsymbol{b}' = -k_2\boldsymbol{n}, \tag{1}$$

where $(\boldsymbol{t})' = \frac{\mathrm{d}(\boldsymbol{t})}{\mathrm{d}s}$, s, k_1 and k_2 are arc coordinate, curvature and torsion respectively of the curve.

In the cross-section of the beam we introduce ξ- and η-directions in coincidence with the principal axes through the centroid O_1, as shown in Fig. 1. The angle between the ξ-axis and normal $\boldsymbol{n}$ is represented as θ, which is generally a function of s. If the unit vectors of $O_1\xi$ and $O_1\eta$ are represented by $\boldsymbol{i}_\xi$ and $\boldsymbol{i}_\eta$, then

$$\begin{aligned} \boldsymbol{i}_\xi &= \boldsymbol{n}\cos\theta + \boldsymbol{b}\sin\theta, \\ \boldsymbol{i}_\eta &= -\boldsymbol{n}\sin\theta + \boldsymbol{b}\cos\theta. \end{aligned} \tag{2}$$

From Eqs. (1) the following expressions are obtained:

$$\begin{aligned} \boldsymbol{t}' &= k_\eta\boldsymbol{i}_\xi - k_\xi\boldsymbol{i}_\eta, \\ \boldsymbol{i}'_\xi &= -k_\eta\boldsymbol{t} + k_\mathrm{s}\boldsymbol{i}_\eta, \\ \boldsymbol{i}'_\eta &= k_\xi\boldsymbol{t} - k_\mathrm{s}\boldsymbol{i}_\xi, \end{aligned} \tag{3}$$

in which $k_\xi = k_1\sin\theta$, $k_\eta = k_1\cos\theta$, $k_\mathrm{s} = k_2 + \theta'$.

The stress–strain relations of the material are given in the local rectangular coordinates in the form (Washizu, 1964):

$$\sigma^{\lambda\mu} = \sigma^{\lambda\mu}(e_{\alpha\beta}), \quad \lambda, \mu, \alpha, \beta = 1, 2, 3, \tag{4}$$

where $\sigma^{\lambda\mu}$ and e_{ij} are the stress and strain tensors defined with respect to the local rectangular coordinates. These tensors are related with those defined with respect to the curvilinear coordinates by the following relationships:

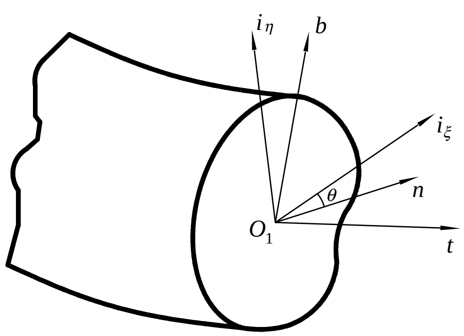

Fig. 1. Geometry of the beam.

$$\sigma^{\lambda\mu} = \frac{\partial y^{\lambda}}{\partial \alpha^{\kappa}} \cdot \frac{\partial y^{\mu}}{\partial \alpha^{\rho}} \tau^{\kappa\rho}, \tag{5}$$

$$e_{\lambda\mu} = \frac{\partial \alpha^{\kappa}}{\partial y^{\lambda}} \cdot \frac{\partial \alpha^{\rho}}{\partial y^{\mu}} f_{\kappa\rho}. \tag{6}$$

We shall denote the components of $\sigma^{\lambda\mu}$ and $e_{\alpha\beta}$ by $(\sigma_{\mathrm{s}}, \sigma_{\xi}, \sigma_{\eta}, \tau_{\mathrm{s}\xi}, \tau_{\mathrm{s}\eta}, \tau_{\xi\eta})$ and $(e_{\mathrm{ss}}, e_{\xi\xi}, e_{\eta\eta}, e_{\mathrm{s}\xi}, e_{\mathrm{s}\eta}, e_{\xi\eta})$, respectively, and the assumptions based on the slenderness of the beam will be employed. The stress components σ_{ξ}, σ_{η} and $\tau_{\xi\eta}$ are assumed to be small compared with those remaining we may put (Washizu, 1964)

$$\sigma_{\xi} = \sigma_{\eta} = \tau_{\xi\eta} = 0, \tag{7}$$

in the stress–strain relations. If the material of the beam is assumed to be isotropic, Eqs. (4) and (7) yield:

$$\sigma_{\mathrm{s}} = Ee_{\mathrm{ss}}, \quad \tau_{\mathrm{s}\xi} = 2Ge_{\mathrm{s}\xi}, \quad \tau_{\mathrm{s}\eta} = 2Ge_{\mathrm{s}\eta}, \tag{8}$$

in which E is Young's modulus of elasticity and G is the shear modulus of the material.

2. Internal forces, equilibrium equations and geometry equations

Simplifying stress vectors to the centroid O_1 on the cross-section A, the principal vector $\boldsymbol{Q}$ and principal moment $\boldsymbol{M}$ can be obtained, of which components are respectively denoted by Q_{s}, Q_{ξ}, Q_{η} and M_{s}, M_{ξ}, M_{η}, so:

$$\boldsymbol{Q} = Q_{\mathrm{s}}\boldsymbol{t} + Q_{\xi}\boldsymbol{i}_{\xi} + Q_{\eta}\boldsymbol{i}_{\eta}, \quad \boldsymbol{M} = M_{\mathrm{s}}\boldsymbol{t} + M_{\xi}\boldsymbol{i}_{\xi} + M_{\eta}\boldsymbol{i}_{\eta},$$

where Q_{s} is axial force, Q_{ξ} and Q_{η} are shear forces, M_{s} is torque, M_{ξ} and M_{η} are bending moments, as shown in Fig. 2. the external forces and moments per unit length along the axis of the beam are indicated by $\boldsymbol{p}$ and $\boldsymbol{m}$ as

$$\boldsymbol{p} = p_{\mathrm{s}}\boldsymbol{t} + p_{\xi}\boldsymbol{i}_{\xi} + p_{\eta}\boldsymbol{i}_{\eta}, \quad \boldsymbol{m} = m_{\mathrm{s}}\boldsymbol{t} + m_{\xi}\boldsymbol{i}_{\xi} + m_{\eta}\boldsymbol{i}_{\eta}.$$

The equilibrium equations are:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\{\boldsymbol{Q}\} - [\boldsymbol{K}] \cdot \{\boldsymbol{Q}\} + \{\boldsymbol{p}\} = \{\boldsymbol{0}\}, \\ &\frac{\mathrm{d}}{\mathrm{d}s}\{\boldsymbol{M}\} - [\boldsymbol{K}] \cdot \{\boldsymbol{M}\} - [\boldsymbol{H}] \cdot \{\boldsymbol{Q}\} + \{\boldsymbol{m}\} = \{\boldsymbol{0}\}, \end{aligned} \tag{9}$$

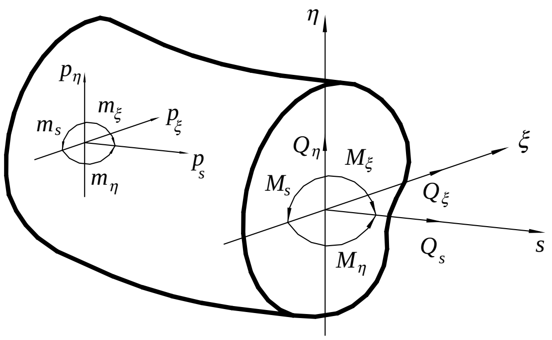

Fig. 2. Stress resultants developed on a typical beam element.

where

$$\{\boldsymbol{Q}\} = [Q_s\ Q_\xi\ Q_\eta]^T, \quad \{\boldsymbol{M}\} = [M_s\ M_\xi\ M_\eta]^T,$$
$$\{\boldsymbol{p}\} = [p_s\ p_\xi\ p_\eta]^T, \quad \{\boldsymbol{m}\} = [m_s\ m_\xi\ m_\eta]^T,$$
$$[\boldsymbol{K}] = \begin{bmatrix} 0 & k_\eta & -k_\xi \\ -k_\eta & 0 & k_s \\ k_\xi & -k_s & 0 \end{bmatrix}, \quad [\boldsymbol{H}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

The general solutions are Hanwei and Peiyuan (1997) and Yuchun et al. (1999):

$$\begin{aligned} \{\boldsymbol{Q}\} &= [\boldsymbol{A}] \cdot \left(\{\boldsymbol{Q}_0\} - \int_0^s [\boldsymbol{A}]^T \cdot \{\boldsymbol{p}\}\,\mathrm{d}s\right), \\ \{\boldsymbol{M}\} &= [\boldsymbol{A}] \cdot \left\{\{\boldsymbol{M}_0\} + \int_0^s [\boldsymbol{A}]^T \cdot ([\boldsymbol{H}] \cdot [\boldsymbol{A}] \cdot (\{\boldsymbol{Q}_0\} + \{\boldsymbol{Q}^*\}) - \{\boldsymbol{m}\}\,\mathrm{d}s\right\}, \end{aligned} \tag{10}$$

where $\{\boldsymbol{Q}_0\}$ and $\{\boldsymbol{M}_0\}$ are integration constants, $\{\boldsymbol{Q}^*\} = -\int_0^s [\boldsymbol{A}]^T \cdot \{\boldsymbol{p}\}\,\mathrm{d}s$.

If the base vectors of special fixed right-handed rectangular coordinate system are $\boldsymbol{i}_x$, $\boldsymbol{i}_y$, $\boldsymbol{i}_z$, then:

$$[\boldsymbol{A}] = \begin{bmatrix} \boldsymbol{t} \cdot \boldsymbol{i}_x & \boldsymbol{t} \cdot \boldsymbol{i}_y & \boldsymbol{t} \cdot \boldsymbol{i}_z \\ \boldsymbol{i}_\xi \cdot \boldsymbol{i}_x & \boldsymbol{i}_\xi \cdot \boldsymbol{i}_y & \boldsymbol{i}_\xi \cdot \boldsymbol{i}_z \\ \boldsymbol{i}_\eta \cdot \boldsymbol{i}_x & \boldsymbol{i}_\eta \cdot \boldsymbol{i}_y & \boldsymbol{i}_\eta \cdot \boldsymbol{i}_z \end{bmatrix}. \tag{11}$$

The geometry equations are:

$$\begin{aligned} &\varepsilon_s = u_s' - k_\eta u_\xi + k_\xi u_\eta, \quad \varepsilon_\xi = u_\xi' + k_\eta u_s - k_s u_\eta - \varphi_\eta, \\ &\varepsilon_\eta = u_\eta' - k_\xi u_s + k_s u_\xi + \varphi_\xi, \quad \omega_s = \varphi_s' - k_\eta \varphi_\xi + k_\xi \varphi_\eta, \\ &\omega_\xi = \varphi_\xi' + k_\eta \varphi_s - k_s \varphi_\eta, \quad \omega_\eta = \varphi_\eta' - k_\xi \varphi_s + k_s \varphi_\xi, \end{aligned} \tag{12}$$

where, ε_s, ε_ξ, ε_η, ω_s, ω_ξ, ω_η are respectively generalized strains corresponding to generalized stresses Q_s, Q_ξ, Q_η, M_s, M_ξ, M_η and u_s, u_ξ, u_η, φ_s, φ_ξ, φ_η are generalized displacements corresponding to loads p_s, p_ξ, p_η, m_s, m_ξ, m_η. The boundary conditions should be given by prescribed the following qualities

$$Q_s \text{ or } u_s, \quad Q_\xi \text{ or } u_\xi, \quad Q_\eta \text{ or } u_\eta, \quad M_s \text{ or } \varphi_s, \quad M_\xi \text{ or } \varphi_\xi, \quad M_\eta \text{ or } u_\eta \tag{13}$$

Eqs. (12) can be rewritten as:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\{\varphi\} - [\boldsymbol{K}] \cdot \{\varphi\} - \{\omega\} = \{\boldsymbol{0}\}, \\ &\frac{\mathrm{d}}{\mathrm{d}s}\{\mathbf{u}\} - [\boldsymbol{K}] \cdot \{\mathbf{u}\} - [\boldsymbol{H}] \cdot \{\varphi\} - \{\varepsilon\} = \{\boldsymbol{0}\}, \end{aligned} \tag{14}$$

where

$$\{\varphi\} = [\varphi_s\ \varphi_\xi\ \varphi_\eta]^T, \quad \{\boldsymbol{u}\} = [u_s\ u_\xi\ u_\eta]^T,$$
$$\{\omega\} = [\omega_s\ \omega_\xi\ \omega_\eta]^T, \quad \{\varepsilon\} = [\varepsilon_s\ \varepsilon_\xi\ \varepsilon_\eta]^T,$$

so the general solutions to the geometry equations are:

$$\begin{aligned} \{\varphi\} &= [\boldsymbol{A}] \cdot (\{\varphi_0\} + \{\varphi^*\}), \\ \{\mathbf{u}\} &= [\boldsymbol{A}] \cdot \left\{\{\boldsymbol{U}_0\} + \int_0^s [\boldsymbol{A}]^T \cdot (\{\varepsilon\} + [\boldsymbol{H}] \cdot [\boldsymbol{A}](\{\varphi_0\} + \{\varphi^*\}))\,\mathrm{d}s\right\}, \end{aligned} \tag{15}$$

in which $\{\varphi_0\}$ and $\{\boldsymbol{U}_0\}$ are integration constants, $\{\varphi^*\} = \int_0^s [\boldsymbol{A}]^T \cdot \{\boldsymbol{\omega}\}\,\mathrm{d}s$.

3. The St.Venant torsional warping function and equivalent constitutive equations

Assuming that the deformations of the beam consist of stretching, bending and torsion,then the displacement field can be written in the following form:

$$\boldsymbol{u} = W\boldsymbol{t} + U\boldsymbol{i}_\xi + V\boldsymbol{i}_\eta, \tag{16}$$

in which:

$$W = u_s(s) + \eta\varphi_\xi(s) - \xi\varphi_\eta(s) + \alpha(s)\varphi(\xi,\eta),$$
$$U = u_\xi(s) - \eta\varphi_s(s), \quad V = u_\eta(s) + \xi\varphi_s(s),$$

here, $\varphi(\xi,\eta)$ is the warping function of the St.Venant torsion of a cylindrical shaft which has the same cross-section as the beam under consideration (Timoshenko and Goodier, 1974). Eq. (16) takes into account the effects of torsion-related warping as well as that of transverse shear deformations. The distributions of strain e_{ss}, $e_{s\xi}$ and $e_{s\eta}$, on the cross-section, are:

$$\begin{aligned}
\sqrt{g}e_{ss} &= \varepsilon_s + \eta\omega_\xi - \xi\omega_\eta + \sqrt{g}\varphi\alpha' + k_s\left[\left(\frac{\partial\varphi}{\partial\xi}\right)\eta - \left(\frac{\partial\varphi}{\partial\eta}\right)\xi\right]\alpha,\\
2\sqrt{g}e_{s\xi} &= \varepsilon_\xi - \eta\omega_s + \left[\sqrt{g}\left(\frac{\partial\varphi}{\partial\xi}\right) + k_\eta\varphi\right]\alpha,\\
2\sqrt{g}e_{s\eta} &= \varepsilon_\eta + \xi\omega_s + \left[\sqrt{g}\left(\frac{\partial\varphi}{\partial\eta}\right) - k_\xi\varphi\right]\alpha,\\
e_{\xi\xi} &= e_{\eta\eta} = e_{\xi\eta} = 0,
\end{aligned} \tag{17}$$

where ε_s, ε_ξ, ε_η, ω_s, ω_ξ, ω_η are the same as Eqs. (12), and α is a generalized coordinate for warping. Assume that the curvature is small enough to assure that (Washizu, 1964):

$$\sqrt{g} \approx 1.$$

Introducing stress resultants and moments defined by:

$$\begin{aligned}
Q_s &= \int\int \sigma_s \,d\xi\,d\eta, \quad M_s = \int\int (\tau_{s\eta}\xi - \tau_{s\xi}\eta)\,d\xi\,d\eta,\\
Q_\xi &= \int\int \tau_{s\xi}\,d\xi\,d\eta, \quad M_\xi = \int\int \sigma_s\eta\,d\xi\,d\eta\\
Q_\eta &= \int\int \tau_{s\eta}\,d\xi\,d\eta, \quad M_\eta = -\int\int \sigma_s\xi\,d\xi\,d\eta,
\end{aligned} \tag{18}$$

lead to the equivalent constitutive equations described with generalized strains and generalized coordinate for warping.

The minimum potential energy principle of the entire beam can be written as

$$\delta U - \delta\int_0^l (\{\boldsymbol{p}\}^T\cdot\{\boldsymbol{u}\} + \{\boldsymbol{m}\}^T\cdot\{\boldsymbol{\varphi}\})\,ds = 0. \tag{19}$$

Equilibrium Eqs. (9) described with generalized displacements can be derived and another equation which involves the St.Venant torsional warping function is:

$$\begin{aligned}
&\left(\int\int \varphi\sigma_s\,d\xi\,d\eta\right)' - k_s\int\int \left[\left(\frac{\partial\varphi}{\partial\xi}\right)\eta - \left(\frac{\partial\varphi}{\partial\eta}\right)\xi\right]\sigma_s\,d\xi\,d\eta\\
&\quad - \int\int \left\{\left[\left(\frac{\partial\varphi}{\partial\xi}\right) + k_\eta\varphi\right]\tau_{s\xi} + \left[\left(\frac{\partial\varphi}{\partial\eta}\right) - k_\xi\varphi\right]\tau_{s\eta}\right\}d\xi\,d\eta = 0.
\end{aligned} \tag{20}$$

Besides, boundary conditions (13) and the following boundary condition can also be obtained

$$\left[\int\int \varphi\sigma_s \,d\xi\, d\eta\right]_0^l = 0. \tag{21}$$

4. The solving process of the generalized coordinate for warping

Equations for the static problem of the beam include: variational Eq. (19), geometry Eqs. (12), and equivalent constitutive Eqs. (18) as well as Eq. (20). Substituting Eqs. (17) into Eqs. (8), then substituting the results obtained into Eq. (20), we have

$$\begin{aligned} &E\int\int \varphi\{\varepsilon_s' + \eta\omega_\xi' - \xi\omega_\eta' + \varphi\alpha'' + k_s' D\alpha + k_s D\alpha'\}\,d\xi\,d\eta - k_s E\int\int D\{\varepsilon_s + \eta\omega_\xi - \xi\omega_\eta + \varphi\alpha' + k_s D\alpha\} \\ &\quad \times d\xi\,d\eta - G\int\int\left[\left(\frac{\partial\varphi}{\partial\xi}\right) + k_\eta\varphi\right]\left\{\varepsilon_\xi - \eta\omega_s + \left[\left(\frac{\partial\varphi}{\partial\xi}\right) + k_\eta\varphi\right]\alpha\right\}d\xi\,d\eta - G\int\int\left[\left(\frac{\partial\varphi}{\partial\eta}\right) - k_\xi\varphi\right] \\ &\quad \times \left\{\varepsilon_\eta + \xi\omega_s + \left[\left(\frac{\partial\varphi}{\partial\eta}\right) - k_\xi\varphi\right]\alpha\right\}d\xi\,d\eta = 0, \end{aligned} \tag{22}$$

where

$$D = \int\int\left[\left(\frac{\partial\varphi}{\partial\xi}\right)\eta - \left(\frac{\partial\varphi}{\partial\eta}\right)\xi\right]d\xi\,d\eta$$

Substituting Eqs. (17) into Eqs. (8), then substituting the results obtained into Eqs. (18), one obtains

$$\begin{aligned} Q_s &= EA\varepsilon_s + Ek_s D\alpha + ES_\xi\omega_\xi - ES_\eta\omega_\eta + E\int\int\varphi\,d\xi\,d\eta\alpha', \\ Q_\xi &= GA\varepsilon_\xi - GS_\xi\omega_s + G\int\int\left[\left(\frac{\partial\varphi}{\partial\xi}\right) + k_\eta\varphi\right]d\xi\,d\eta\alpha, \\ Q_\eta &= GA\varepsilon_\eta + GS_\eta\omega_s + G\int\int\left[\left(\frac{\partial\varphi}{\partial\eta}\right) - k_\xi\varphi\right]d\xi\,d\eta\alpha, \\ M_s &= GS_\eta\varepsilon_\eta - GS_\xi\varepsilon_\xi + GI_P\omega_s - G\int\int\left[\left(\frac{\partial\varphi}{\partial\xi}\right)\eta - \left(\frac{\partial\varphi}{\partial\eta}\right)\xi + \xi k_\xi\varphi + \eta k_\eta\varphi\right]d\xi\,d\eta\alpha, \\ M_\xi &= ES_\xi\varepsilon_s + EI_\xi\omega_\xi - EI_{\xi\eta}\omega_\eta + Ek_s\int\int\left[\left(\frac{\partial\varphi}{\partial\xi}\right)\eta^2 - \left(\frac{\partial\varphi}{\partial\eta}\right)\xi\eta\right]d\xi\,d\eta\alpha + E\int\int\eta\varphi\,d\xi\,d\eta\alpha', \\ M_\eta &= -ES_\eta\varepsilon_s + EI_\eta\omega_\eta - EI_{\xi\eta}\omega_\xi - Ek_s\int\int\left[\left(\frac{\partial\varphi}{\partial\xi}\right)\xi\eta - \left(\frac{\partial\varphi}{\partial\eta}\right)\xi^2\right]d\xi\,d\eta\alpha - E\int\int\xi\varphi\,d\xi\,d\eta\alpha', \end{aligned} \tag{23}$$

in which, A is the area of the cross-section, S_ξ and S_η are the first moments of the cross-sectional area with respect to the ξ- and η-axes, I_ξ and I_η are the moments of inertia of the cross-sectional area, computed about the ξ- and η-axes, $I_{\xi\eta}$ is the product of inertia, and $I_p = I_\xi + I_\eta$. Six unknown generalized strains can be now obtained from Eqs. (23), implying

$$\begin{aligned}
\varepsilon_s &= \frac{1}{E}(D_{11}Q_s + D_{12}M_\xi + D_{13}M_\eta) + k_s H_1 \alpha + H_2 \alpha', \\
\varepsilon_\xi &= \frac{1}{G}(D_{21}Q_\xi + D_{22}Q_\eta + D_{23}M_s) + H_3 \alpha, \\
\varepsilon_\eta &= \frac{1}{G}(D_{22}Q_\xi + D_{32}Q_\eta + D_{33}M_s) + H_4 \alpha, \\
\omega_s &= \frac{1}{G}(D_{23}Q_\xi + D_{33}Q_\eta + D_{43}M_s) + H_5 \alpha, \\
\omega_\xi &= \frac{1}{G}(D_{12}Q_s + D_{52}M_\xi + D_{53}M_\eta) + k_s H_6 \alpha + H_7 \alpha', \\
\omega_\eta &= \frac{1}{G}(D_{13}Q_s + D_{53}M_\xi + D_{63}M_s) + k_s H_8 \alpha + H_9 \alpha',
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
&D_{11} = \frac{I_{\xi\eta}^2 - I_\xi I_\eta}{D_1}, \quad D_{12} = \frac{I_\eta S_\xi - I_{\xi\eta} S_\eta}{D_1}, \quad D_{13} = \frac{I_{\xi\eta} S_\xi - I_\xi S_\eta}{D_1}, \quad D_{21} = \frac{S_\eta^2 - I_P A}{D_2}, \\
&D_{22} = \frac{S_\xi S_\eta}{D_2}, \quad D_{23} = -\frac{S_\xi A}{D_2}, \quad D_{32} = \frac{S_\xi^2 - I_P A}{D_2}, \quad D_{33} = \frac{S_\eta A}{D_2}, \\
&D_{43} = -\frac{A^2}{D_2}, \quad D_{52} = \frac{S_\eta^2 - I_\eta A}{D_1}, \quad D_{53} = \frac{-I_{\xi\eta} A + S_\xi S_\eta}{D_1}, \quad D_{63} = \frac{S_\xi^2 - I_\xi A}{D_1}, \\
&D_1 = (I_{\xi\eta}^2 A - 2 I_{\xi\eta} S_\xi S_\eta - I_\xi I_\eta A + I_\eta S_\xi^2 + I_\xi S_\eta^2), \\
&D_2 = -A(I_P A - S_\xi^2 - S_\eta^2), \\
&H_1 = [(I_\xi I_\eta - I_{\xi\eta}^2) D + (I_{\xi\eta} S_\eta - I_\eta S_\xi) J_7 + (I_{\xi\eta} S_\xi - I_\xi S_\eta) J_8]/D_1, \\
&H_2 = [(I_\xi I_\eta - I_{\xi\eta}^2) J_1 + (I_{\xi\eta} S_\eta - I_\eta S_\xi) J_5 + (I_{\xi\eta} S_\xi - I_\xi S_\eta) J_6]/D_1, \\
&H_3 = [(I_P A - S_\eta^2) J_2 - S_\xi S_\eta J_3 - S_\xi A J_4]/D_2, \\
&H_4 = [-S_\xi S_\eta J_2 + (I_P A - S_\xi^2) J_3 + S_\eta A J_4]/D_2, \\
&H_5 = (S_\xi A J_2 - S_\eta A J_3 - A^2 J_4)/D_2, \\
&H_6 = [(I_{\xi\eta} S_\eta - I_\eta S_\xi) D + (I_\eta A - S_\eta^2) J_7 + (S_\xi S_\eta - I_{\xi\eta} A) J_8]/D_1, \\
&H_7 = [(I_{\xi\eta} S_\eta - I_\eta S_\xi) J_1 + (I_\eta A - S_\eta^2) J_5 + (S_\xi S_\eta - I_{\xi\eta} A) J_6]/D_1, \\
&H_8 = [(I_\xi S_\eta - I_{\xi\eta} S_\xi) D + (I_{\xi\eta} A - S_\xi S_\eta) J_7 + (S_\xi^2 - I_\xi A) J_8]/D_1, \\
&H_9 = [(I_\xi S_\eta - I_{\xi\eta} S_\xi) J_1 + (I_{\xi\eta} A - S_\xi S_\eta) J_5 + (S_\xi^2 - I_\xi A) J_6]/D_1.
\end{aligned}$$

where, $J_i (i = 1, 2, \ldots, 8)$, which depend only on the curvature and geometry of the beam, are defined as the following integrations given by

$$\begin{aligned}
&J_1 = \int\int \varphi \, d\xi \, d\eta, \quad J_2 = \int\int \left[\left(\frac{\partial \varphi}{\partial \xi} \right) + k_\eta \varphi \right] d\xi \, d\eta, \quad J_3 = \int\int \left[\left(\frac{\partial \varphi}{\partial \eta} \right) - k_\xi \varphi \right] d\xi \, d\eta, \\
&J_4 = \int\int \left[\left(\frac{\partial \varphi}{\partial \xi} \right) \eta - \left(\frac{\partial \varphi}{\partial \eta} \right) \xi + \eta k_\eta \varphi + \xi k_\xi \varphi \right] d\xi \, d\eta, \quad J_5 = \int\int \eta \varphi \, d\xi \, d\eta, \quad J_6 = \int\int \xi \varphi \, d\xi \, d\eta, \\
&J_7 = \int\int \left[\left(\frac{\partial \varphi}{\partial \xi} \right) \eta^2 - \left(\frac{\partial \varphi}{\partial \eta} \right) \xi \eta \right] d\xi \, d\eta, \quad J_8 = \int\int \left[\left(\frac{\partial \varphi}{\partial \xi} \right) \xi \eta - \left(\frac{\partial \varphi}{\partial \eta} \right) \xi^2 \right] d\xi \, d\eta.
\end{aligned}$$

Substituting Eqs. (24) and their derivatives ε_s', ω_ξ', ω_η' into Eq. (22), we establish the differential equation of α:

$$\begin{aligned}
&\alpha'' + \frac{k_s(H_1J_1 + H_6J_5 - H_8J_6 - H_2D - H_7J_7 + H_9J_8)}{(H_2J_1 + H_7J_5 - H_9J_6 + \Gamma)}\alpha' + \frac{1}{(H_2J_1 + H_7J_5 - H_9J_6 + \Gamma)} \\
&\times \left\{ k_s'H_1J_1 + k_s'H_6J_5 - k_s'H_8J_6 + k_s'\int\int \varphi D \, d\xi \, d\eta - k_s^2H_1D - k_s^2H_6J_7 + k_s^2H_8J_8 - k_s^2 \int\int D^2 \, d\xi \, d\eta \right\} \\
&\times \alpha - \frac{G}{E(H_2J_1 + H_7J_5 - H_9J_6 + \Gamma)} \left\{ H_3J_2 + H_4J_3 + H_5J_4 + \int\int \left[\left(\frac{\partial\varphi}{\partial\xi} \right) + k_\eta\varphi \right]^2 d\xi \, d\eta \right. \\
&\left. + \int\int \left[\left(\frac{\partial\varphi}{\partial\eta} \right) - k_\xi\varphi \right]^2 d\xi \, d\eta \right\} \alpha + \frac{1}{E(H_2J_1 + H_7J_5 - H_9J_6 + \Gamma)} [D_{11}J_1 + D_{12}J_5 - D_{13}J_6)Q_s' \\
&+ (D_{12}J_1 + D_{52}J_5 - D_{53}J_6)M_\xi' + (D_{13}J_1 + D_{53}J_5 - D_{63}J_6)M_\eta' - k_s(D_{11}D + D_{12}J_7 - D_{13}J_8)Q_s \\
&- (D_{21}J_2 + D_{22}J_3 - D_{23}J_4)Q_\xi - (D_{22}J_2 + D_{32}J_3 + D_{33}J_4)Q_\eta + (D_{23}J_2 - D_{33}J_3 + D_{43}J_4)M_s \\
&- k_s(D_{12}D + D_{52}J_7 - D_{53}J_8)M_\xi - k_s(D_{13}D + D_{53}J_7 - D_{63}J_8)M_\eta] = 0,
\end{aligned} \tag{25}$$

where

$$\Gamma = \int\int \varphi^2 \, d\xi \, d\eta.$$

Eq. (25) can be written in the form:

$$\alpha'' + q_1(s)\alpha' + q_2(s)\alpha = f(s). \tag{26}$$

This is a second order ordinary non-homogeneous linear differential equation with variable coefficients. The solution to such an equation is the sum of two functions α_c, the complementary solution to the homogeneous equation,

$$\alpha_c'' + q_1(s)\alpha_c' + q_2(s)\alpha_c = 0, \tag{27}$$

and any particular solution α^* to

$$(\alpha^*)'' + q_1(s)(\alpha^*)' + q(s)\alpha^* = f(s). \tag{28}$$

Once the relation among k_1, k_2 and arc coordinate s is determined, we may find its solution. The general solving steps are as follows:

Let $\alpha_1(s)$ and $\alpha_2(s)$ be any two linear independent solutions to the homogeneous Eq. (27), then the complete solution to Eq. (26) is:

$$\alpha = c_1\alpha_1(s) + c_2\alpha_2(s) + \alpha^*, \tag{29}$$

in which α^* is a particular solution to Eq. (26). Thus, $\{\omega\}$ and $\{\varepsilon\}$ can be expressed with $\{\boldsymbol{Q}\}$, $\{\boldsymbol{M}\}$, the generalized coordinate for warping α and integration constants c_1, c_2. Now the rest is to determine integration constants $\{\boldsymbol{Q}_0\}$, $\{\boldsymbol{M}_0\}$, $\{\boldsymbol{\varphi}_0\}$, $\{\boldsymbol{U}_0\}$ as well as c_1 and c_2.

5. The generalized coordinate for warping of a plane curved beam under a concentrated force at the free end of the beam and two uniformly distributed loads

$k_s = 0$ and $\theta = 5\pi/6$ in Eqs. (3) is just the situation of a plane curved beam (see Fig. 3a), and Fig. 3b illustrates the uniform equilateral triangle cross-section at the free end of the beam. It should be note that ξ- and η-axes are a set of principal axes through the centroid on the cross-section. Fix the origin of the rect-

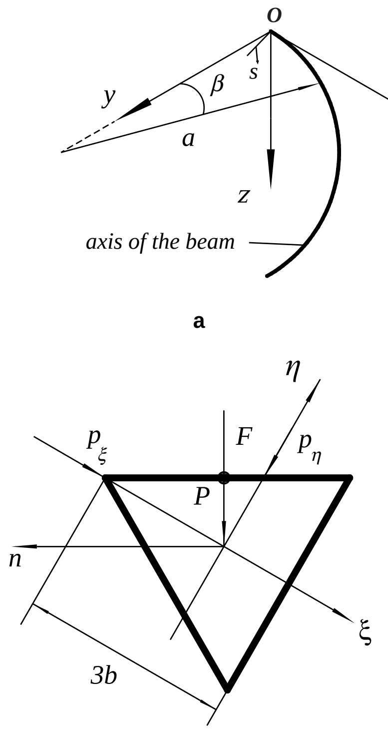

Fig. 3. A plane curved beam with equilateral triangle cross-section: (a) Axis of a plane curved beam and (b) cross-section at the free end of the beam.

angular coordinate system at the end of the beam ($s = 0$), the axis of the beam being on the plane Oxy. The loads acting are:

$$\{\boldsymbol{m}\} = \{\boldsymbol{0}\}, \quad \{\boldsymbol{p}\} = [0\ p_\xi\ p_\eta]^{\mathrm{T}}$$

If the axis of the beam is a circle with radius a, one has:

$$\beta = \frac{s}{a}, \quad k_\xi = k_1 \sin\theta, \quad k_\eta = k_1\cos\theta,$$
$$x = a\sin\beta, \quad y = a(1-\cos\beta).$$

Eq. (25) can be reduced to:

$$\alpha'' - \frac{G}{E(\Gamma - \frac{J_5^2}{I_\xi})}\left\{\int\int\left[\left(\frac{\partial\varphi}{\partial\xi}\right) + k_\eta\varphi\right]^2 \mathrm{d}\xi\,\mathrm{d}\eta + \int\int\left[\left(\frac{\partial\varphi}{\partial\eta}\right) - k_\xi\varphi\right]^2 \mathrm{d}\xi\,\mathrm{d}\eta - \frac{J_3^2}{A} - \frac{J_4^2}{I_P}\right\}\alpha$$
$$= f_1 Q_\xi + f_2 M_{\mathrm{s}}, \tag{30}$$

where

$$f_1 = \frac{\left(\frac{J_5}{I_\xi} - \frac{J_3}{A}\right)}{E\left(\Gamma - \frac{J_5^2}{I_\xi}\right)}, \quad f_2 = \frac{\left(\frac{J_4}{I_P} - k_\eta\frac{J_5}{I_\xi}\right)}{E\left(\Gamma - \frac{J_5^2}{I_\xi}\right)}$$

The solution to Eq. (30) must be:

$$\alpha = c_1 ch\bar{\alpha}s + c_2 sh\bar{\alpha}s + \alpha^* \tag{31}$$

where

$$\bar{\alpha} = \sqrt{G\left\{\int\int\left[\left(\frac{\partial\varphi}{\partial\xi}\right) + k_\eta\varphi\right]^2 \mathrm{d}\xi\,\mathrm{d}\eta + \int\int\left[\left(\frac{\partial\varphi}{\partial\eta}\right) - k_\xi\varphi\right]^2 \mathrm{d}\xi\,\mathrm{d}\eta - \frac{J_3^2}{A} - \frac{J_4^2}{I_P}\right\}\Big/ E\left(\Gamma - \frac{J_5^2}{I_\xi}\right)}$$

Thus, the complete solution of Eq. (30) becomes:

$$\begin{aligned}\alpha &= c_1 ch\bar{\alpha}s + c_2 sh\bar{\alpha}s \\ &\quad + \frac{f_1}{(k_1^2+\bar{\alpha}^2)}\left\{\frac{1}{2}Q_{0s}\sin\beta - \frac{1}{2}Q_{0\xi}\cos\beta - \frac{\sqrt{3}}{2\bar{\alpha}^2}Q_{0\eta}(k_1^2+\bar{\alpha}^2) + \frac{\sqrt{3}}{4}p_\xi a\left[\frac{s}{a\bar{\alpha}^2}(k_1^2+\bar{\alpha}^2) - \sin\beta\right]\right. \\ &\quad \left. - \frac{1}{4}p_\eta a\left[\frac{3s}{a\bar{\alpha}^2}(k_1^2+\bar{\alpha}^2) + \sin\beta\right]\right\} + \frac{f_2}{(k_1^2+\bar{\alpha}^2)}\left\{M_{0s}\cos\beta + M_{0\xi}\sin\beta - Q_{0\eta}a\left[\cos\beta - \frac{1}{\bar{\alpha}^2}(k_1^2+\bar{\alpha}^2)\right]\right. \\ &\quad \left. + \frac{1}{2}p_\xi a^2\left[\sin\beta - \frac{s}{a\bar{\alpha}^2}(k_1^2+\bar{\alpha}^2)\right] - \frac{\sqrt{3}}{2}p_\eta a^2\left[\sin\beta - \frac{s}{a\bar{\alpha}^2}(k_1^2+\bar{\alpha}^2)\right]\right\}.\end{aligned} \tag{32}$$

Using Eqs. (10) and (15), we have

$$\begin{aligned}
M_s &= M_{0s}\cos\beta + M_{0\xi}\sin\beta + Q_{0\eta}y + \frac{1}{2}p_\xi a^2(\sin\beta - \beta) - \frac{\sqrt{3}}{2}p_\eta a^2(\sin\beta-\beta),\\
M_\xi &= \frac{\sqrt{3}}{2}M_{0s}\sin\beta - \frac{\sqrt{3}}{2}M_{0\xi}\cos\beta + \frac{1}{2}M_{0\eta} + \frac{1}{2}Q_{0s}y - \frac{1}{2}Q_{0\xi}x - \frac{\sqrt{3}}{2}Q_{0\eta}x - p_\eta ay,\\
M_\eta &= \frac{1}{2}M_{0s}\sin\beta - \frac{1}{2}M_{0\xi}\cos\beta - \frac{\sqrt{3}}{2}M_{0\eta} - \frac{\sqrt{3}}{2}Q_{0s}y + \frac{\sqrt{3}}{2}Q_{0\xi}x - \frac{1}{2}Q_{0\eta}x + p_\xi ay,\\
Q_s &= Q_{0s}\cos\beta + Q_{0\xi}\sin\beta + \frac{\sqrt{3}}{2}p_\xi y + \frac{1}{2}p_\eta y,\\
Q_\xi &= \frac{\sqrt{3}}{2}Q_{0s}\sin\beta - \frac{\sqrt{3}}{2}Q_{0\xi}\cos\beta + \frac{1}{2}Q_{0\eta} - \frac{1}{4}p_\xi a(3\sin\beta+\beta) - \frac{\sqrt{3}}{4}p_\eta a(\sin\beta-\beta),\\
Q_\eta &= \frac{1}{2}Q_{0s}\sin\beta - \frac{1}{2}Q_{0\xi}\cos\beta - \frac{\sqrt{3}}{2}Q_{0\eta} - \frac{\sqrt{3}}{4}p_\xi a(\sin\beta-\beta) - \frac{1}{4}p_\eta a(\sin\beta+3\beta),\\
\varphi_s &= \varphi_{0s}\cos\beta + \varphi_{0\xi}\sin\beta + a\cos\beta\int_0^\beta\left(\omega_s\cos\beta + \frac{\sqrt{3}}{2}\omega_\xi\sin\beta + \frac{1}{2}\omega_\eta\sin\beta\right)\mathrm{d}\beta\\
&\quad + a\sin\beta\int_0^\beta\left(\omega_s\sin\beta - \frac{\sqrt{3}}{2}\omega_\xi\cos\beta - \frac{1}{2}\omega_\eta\cos\beta\right)\mathrm{d}\beta,\\
\varphi_\xi &= \frac{\sqrt{3}}{2}\varphi_{0s}\sin\beta - \frac{\sqrt{3}}{2}\varphi_{0\xi}\cos\beta + \frac{1}{2}\varphi_{0\eta}\\
&\quad + \frac{\sqrt{3}}{2}a\sin\beta\int_0^\beta\left(\omega_s\cos\beta + \frac{\sqrt{3}}{2}\omega_\xi\sin\beta + \frac{1}{2}\omega_\eta\sin\beta\right)\mathrm{d}\beta
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\sqrt{3}}{2}a\cos\beta\int_0^\beta\left(\omega_s\sin\beta-\frac{\sqrt{3}}{2}\omega_\xi\cos\beta-\frac{1}{2}\omega_\eta\cos\beta\right)d\beta+\frac{1}{4}a\int_0^\beta\left(\omega_\xi-\sqrt{3}\omega_\eta\right)d\beta,\\
\varphi_\eta=&\frac{1}{2}\varphi_{0s}\sin\beta-\frac{1}{2}\varphi_{0\xi}\cos\beta-\frac{\sqrt{3}}{2}\varphi_{0\eta}+\frac{1}{2}a\sin\beta\int_0^\beta\left(\omega_s\cos\beta+\frac{\sqrt{3}}{2}\omega_\xi\sin\beta+\frac{1}{2}\omega_\eta\sin\beta\right)d\beta\\
&-\frac{1}{2}a\cos\beta\int_0^\beta\left(\omega_s\sin\beta-\frac{\sqrt{3}}{2}\omega_\xi\cos\beta-\frac{1}{2}\omega_\eta\cos\beta\right)d\beta-\frac{\sqrt{3}}{4}a\int_0^\beta(\omega_\xi-\sqrt{3}\omega_\eta)\,d\beta,\\
u_s=&U_{0s}\cos\beta+U_{0\xi}\sin\beta+a\cos\beta\int_0^\beta\left[\varepsilon_s\cos\beta-\frac{\sqrt{3}}{2}\sin\beta(\varepsilon_\xi+\varphi_\eta)-\frac{1}{2}\sin\beta(\varepsilon_\eta-\varphi_\xi)\right]d\beta\\
&+a\sin\beta\int_0^\beta\left[\varepsilon_s\sin\beta-\frac{\sqrt{3}}{2}\cos\beta(\varepsilon_\xi+\varphi_\eta)+\frac{1}{2}\cos\beta(\varepsilon_\eta-\varphi_\xi)\right]d\beta,\\
u_\xi=&\frac{\sqrt{3}}{2}U_{0s}\sin\beta-\frac{\sqrt{3}}{2}U_{0\xi}\cos\beta+\frac{1}{2}U_{0\eta}+\frac{\sqrt{3}}{2}a\sin\beta\\
&\times\int_0^\beta\left[\varepsilon_s\cos\beta+\frac{\sqrt{3}}{2}\sin\beta(\varepsilon_\xi+\varphi_\eta)+\frac{1}{2}\sin\beta(\varepsilon_\eta-\varphi_\xi)\right]d\beta\\
&-\frac{\sqrt{3}}{2}a\cos\beta\int_0^\beta\left[\varepsilon_s\sin\beta-\frac{\sqrt{3}}{2}\cos\beta(\varepsilon_\xi+\varphi_\eta)-\frac{1}{2}\cos\beta(\varepsilon_\eta-\varphi_\xi)\right]d\beta\\
&+\frac{1}{4}a\int_0^\beta[(\varepsilon_\xi+\varphi_\eta)-\sqrt{3}(\varepsilon_\eta-\varphi_\xi)]\,d\beta,\\
u_\eta=&\frac{1}{2}U_{0s}\sin\beta-\frac{1}{2}U_{0\xi}\cos\beta-\frac{\sqrt{3}}{2}U_{0\eta}\\
&+\frac{1}{2}a\sin\beta\int_0^\beta\left[\varepsilon_s\cos\beta+\frac{\sqrt{3}}{2}\sin\beta(\varepsilon_\xi+\varphi_\eta)-\frac{1}{2}\sin\beta(\varepsilon_\eta-\varphi_\xi)\right]d\beta\\
&-\frac{1}{2}a\cos\beta\int_0^\beta\left[\varepsilon_s\sin\beta-\frac{\sqrt{3}}{2}\cos\beta(\varepsilon_\xi+\varphi_\eta)-\frac{1}{2}\cos\beta(\varepsilon_\eta-\varphi_\xi)\right]d\beta\\
&+\frac{\sqrt{3}}{4}a\int_0^\beta\left[(\varepsilon_\xi+\varphi_\eta)-\sqrt{3}(\varepsilon_\eta-\varphi_\xi)\right]d\beta.
\end{aligned}
\tag{33}
$$

If the beam is fixed at $s = 0$ and loaded by a concentrated force F applied at the other end ($s = l$) in the vertical direction (see Fig. 3b). The boundary conditions are:

$$
\begin{aligned}
&s=0(\beta=0),\quad U_{0s}=U_{0\xi}=U_{0\eta}=0,\quad \varphi_{0s}=\varphi_{0\xi}=\varphi_{0\eta}=0,\quad \alpha=0,\\
&s=l(\beta=\beta_l),\quad Q_s=0,\quad Q_\xi=\frac{1}{2}F,\quad Q_\eta=\frac{\sqrt{3}}{2}F,\quad M_s=M_\xi=M_\eta=0,\quad \alpha'=0,
\end{aligned}
$$

where $l = \pi a/2$, the integration constants determined by the aforementioned conditions are

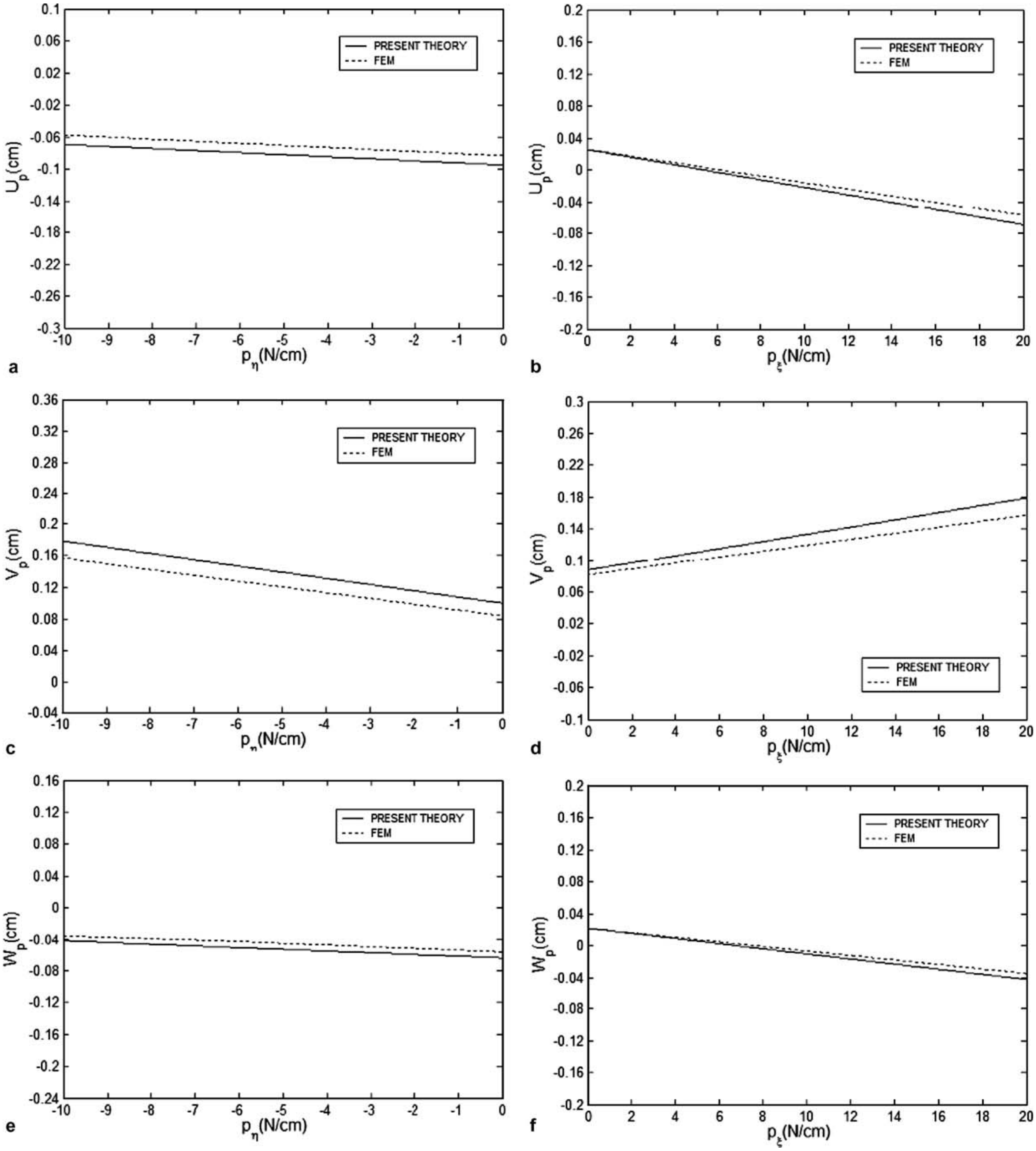

Fig. 4. (a) Horizontal displacement U_P of point P at the free end of the beam under a concentrated force $F = 25$ N applied at the free end and uniformly distributed loads $p_\xi = 20$ N/cm, p_η. (b) Horizontal displacement U_P of point P at the free end of the beam under a concentrated force $F = 25$ N applied at the free end and uniformly distributed loads p_ξ, $p_\eta = 10$ N/cm. (c) Vertical displacement V_P of point P at the free end of the beam under a concentrated force $F = 25$ N applied at the free end and uniformly distributed loads $p_\xi = 20$ N/cm, p_η. (d) Vertical displacement V_P of point P at the free end of the beam under a concentrated force $F = 25$ N applied at the free end of and uniformly distributed loads p_ξ, $p_\eta = 10$ N/cm. (e) Warping displacement W_P of point P at the free end of the beam under a concentrated force $F = 25$ N applied at the free end and uniformly distributed loads $p_\xi = 20$ N/cm, p_η. (f) Warping displacement W_P of point P at the free end of the beam under a concentrated force $F = 25$ N applied at the free end and uniformly distributed loads p_ξ, $p_\eta = 10$ N/cm.

$$Q_{0s} = \frac{\sqrt{3}}{2}p_\xi a + \frac{1}{2}p_\eta a + \frac{1}{2}F, \quad Q_{0\xi} = -\frac{\sqrt{3}}{2}p_\xi a - \frac{1}{2}p_\eta a, \quad Q_{0\eta} = \frac{\pi}{4}p_\xi a - \frac{\sqrt{3}\pi}{4}p_\eta a + F,$$

$$M_{0s} = -\frac{1}{2}p_\xi a^2\left(1-\frac{\pi}{2}\right) + \frac{\sqrt{3}}{2}p_\eta a^2\left(1-\frac{\pi}{2}\right) + Fa, \quad M_{0\xi} = -\frac{1}{2}p_\xi a^2 + \frac{\sqrt{3}}{2}p_\eta a^2 - Fa,$$

$$M_{0\eta} = -\frac{\sqrt{3}}{2}p_\xi a^2 - \frac{1}{2}p_\eta a^2,$$

$$c_1 = \frac{f_1}{(k_1^2+\bar{\alpha}^2)}\left[\frac{1}{2}Q_{0\xi} + \frac{\sqrt{3}}{2\bar{\alpha}^2}Q_{0\eta}(k_1^2+\bar{\alpha}^2)\right] - \frac{f_2}{(k_1^2+\bar{\alpha}^2)}\left(M_{0s} + \frac{k_1}{\bar{\alpha}^2}Q_{0\eta}\right),$$

$$c_2 = -c_1\,\mathrm{th}(\bar{\alpha}l) - \frac{f_1}{\bar{\alpha}ch\bar{\alpha}l(k_1^2+\bar{\alpha}^2)}\left[\frac{1}{2a}Q_{0\xi} + \frac{\sqrt{3}}{4\bar{\alpha}^2}p_\xi(k_1^2+\bar{\alpha}^2) - \frac{3}{4\bar{\alpha}^2}p_\eta(k_1^2+\bar{\alpha}^2)\right]$$
$$+ \frac{f_2}{\bar{\alpha}ch\bar{\alpha}l(k^2+\bar{\alpha}^2)}\left[\frac{1}{a}M_{0s} - Q_{0\eta} + \frac{1}{2\bar{\alpha}^2}p_\xi a(k_1^2+\bar{\alpha}^2) + \frac{\sqrt{3}}{2\bar{\alpha}^2}p_\eta a(k_1^2+\bar{\alpha}^2)\right]. \tag{34}$$

So far, the solutions to this problem have been obtained. The beam in Fig. 3 is made of steel with the following properties

$$E = 2.106\times10^5\ \text{MPa}, \quad G = 0.816\times10^5\ \text{MPa},$$
$$b = 1.732\ \text{cm}, \quad a = 48\ \text{cm}.$$

The St. Venant torsional warping function (Sokolnikoff, 1956)

$$\varphi = -\frac{1}{6b}(\eta^3 - 3\xi^2\eta) = -\frac{1}{10.392}(\eta^3 - 3\xi^2\eta).$$

Let U_P and V_P present the displacements in the horizontal and vertical directions of point P on the cross-section ($\beta = \frac{\pi}{2}$) shown in Fig. 3b, respectively, and W_P present the displacement in the s direction of point P. theoretical results for U_P, V_P and W_P are obtained using the equations developed in this paper and compared with a 3-D finite element analysis (referred as the FEM results), according to the ANSYS program.

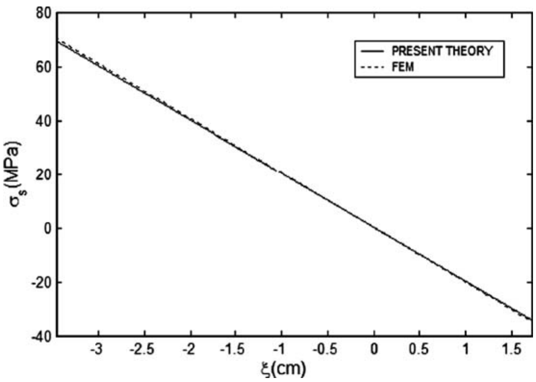

Fig. 5. Distribution of the axial stress σ_s on ξ-axis at the root of the beam under uniformly distributed loads $p_\xi = 20$ N/cm, $p_\eta = 10$ N/cm and a concentrated force $F = 25$ N applied at the free end.

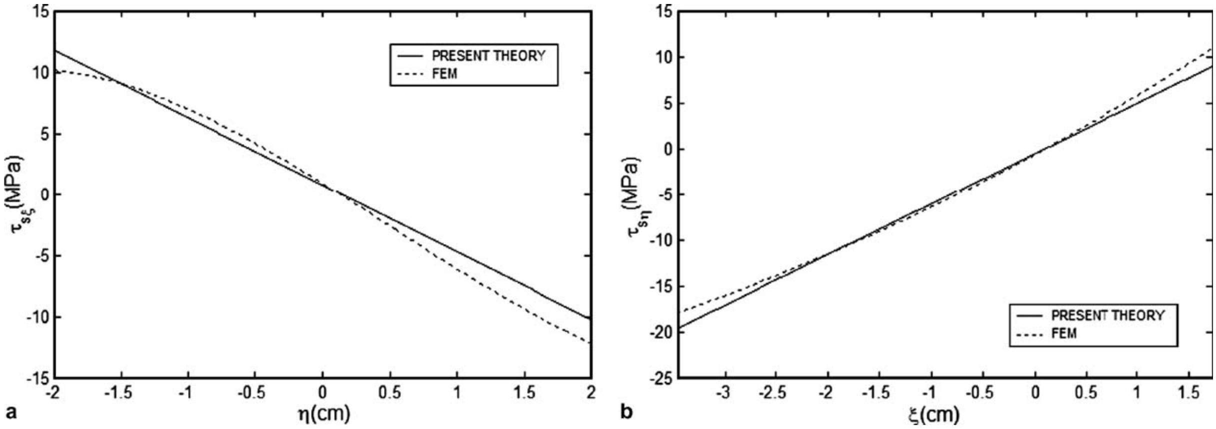

Fig. 6. (a) Distribution of shear stress $\tau_{s\xi}$ on ξ-axis at the root of the beam under uniformly distributed loads $p_\xi = 20$ N/cm, $p_\eta = 10$ N/cm and a concentrated force $F = 25$ N applied at the free end. (b). Distribution of the shear stress $\tau_{s\eta}$ on ξ-axis at the root of the beam under uniformly distributed loads $p_\xi = 20$ N/cm, $p_\eta = 10$ N/cm and a concentrated force $F = 25$ N applied at the free end.

To analyze the beam shown in Fig. 3 by the finite element method (FEM), we partition it into 35299 three-dimensional solid elements (SOLID 92), and the total number of nodal points is 54482. These cases for different values of p_ξ and p_η are shown in Fig. 4(a)–(f). It is evident that the theoretical results are very close to the FEM results.

It is also interesting to compute the stress distributions at the root ($\beta = 0$) of the beam. Fig. 5 shows the distribution of the axial stress σ_s on the ξ-axis at the root of the beam under uniformly distributed loads $p_\xi = 20$ N/cm, $p_\eta = 10$ N/cm and a concentrated force $F = 25$ N applied at the free end. Here, the axial stress distribution is in close agreement between theoretical results and the FEM results. While Fig. 6 shows the distributions of the shear stresses $\tau_{s\xi}$ and $\tau_{s\eta}$ on the ξ-axis at the root of the beam, respectively.

6. Conclusions

A differential equation of generalized warping coordinate α for naturally curved and twisted beams with general cross-sectional shapes subjected to arbitrary load has been derived. The numerical results obtained by solving the present equations are very close to the FEM results for different values of p_ξ and p_η. The highlights of this paper are as follows:

(i) From mathematical point of view, the boundary value problem for the theory of the beams is fully defined in Eqs. (12), (13), (18) and (29). The complex structural behavior of the beams is modeled accurately as the following features are included shear deformations and torsion-related warping.
(ii) It should be emphasized that this theory is not limited to the beams with an equilateral triangle cross-section, it can be extended to those with general cross-sectional shapes. Once the generalized coordinate for warping α is found, the solutions (23) and (24) remain valid.
(iii) This theory is not limited to the beams with solid cross section either. In the case of thin-walled special curved beams made of either isotropic or composite materials, it can also be extended as long as several assumptions given by Washizu (1964) remain valid. Of course, the stress–strain (constitutive) relations for the beams with thin-walled cross-sections should be supplied.the constitutive equations, providing an additional information about the material and geometrical properties of the body under consideration, complete the formulation of the boundary value problem for the 1-D theory.

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